International Journal of Advanced and Applied Sciences, 5(10) 2018, Pages: 28-34

Contents lists available at Science-Gate

International Journal of Advanced and Applied Sciences

Journal homepage: http://www.science-gate.com/IJAAS.html

Caputo MSM fractional differentiation of extended Mittag-Leffler function

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ARTICLE INFO

Article history:
Received 28 April 2018
Received in revised form
29 July 2018
Accepted 5 August 2018

Keywords:
Extended Mittag-Leffler function
Caputo MSM fractional different-ion
Hadmard product

ABSTRACT

Recently, many researchers are interested in the investigation of an extended form of special functions like Gamma function, Beta function, Gauss hypergeometric function, Confluent hypergeometric function and Mittag-Leffler function etc. Here, in this paper, the main objective is to find the composition of Caputo MSM fractional differential of the extended form of Mittag-Leffler function in terms of extended Beta function. Further, in this sequel, some corollaries and consequences are shown that are the special case of our main findings.

1. Introduction

Integral and differential operators in fractional calculus have become research subject in recently few decades due to ability of having arbitrary order. For more recent developments in fractional integral and differential operators, we refer the reader to see Agarwal and Choi (2016), Choi and Agarwal (2014), Choi and Agarwal (2015), Gehlot (2013), Gupta and Parihar (2017), Kilbas et al. (2004), Nadir et al. (2014), Rahman et al. (2017b), Saxena and Parmar (2017), Shishkina and Sitnik (2017), Singh (2013), Srivastava and Agarwal (2013), Srivaastava et al. (2012), Suthar et al. (2017), and the references cited therein.

Now a day, a general trend is in the extensions of special functions like Gamma function, Beta function, Gauss hypergeometric function and Mittag-Leffler function etc. due to its diverse applications in many applied fields. One can consult the papers by Chaudhry et al. (1997, 2004), Luo and Raina (2013), Özarslan and Yilmaz (2014), Rahman et al. (2017b), and Srivastava et al. (2012) containing the bibliography therein.

Srivastava et al. (2012) defined a function

$$\Theta\left(\{\kappa_n\}_{n\epsilon N_0}; z\right) := \begin{cases} \sum_{n=0}^{\infty} \kappa_n \frac{z^n}{n!} & \begin{pmatrix} |z| < \Re \\ 0 < \Re < \infty \\ \kappa_0 := 1 \end{pmatrix} \\ \mathfrak{m}_0 z^{\varpi} \exp(z)\left[1 + O\left(\frac{1}{z}\right)\right] & \begin{pmatrix} \Re(z) \to \infty \\ \mathfrak{m}_0 > 0; \varpi \epsilon C \end{pmatrix} \end{cases}$$

where $\Theta\left(\{\kappa_n\}_{n\epsilon N_0}; z\right)$ is considered to be analytical within $|z| < \Re, 0 < \Re < \infty$ and $\{\kappa_n\}_{n\epsilon N_0}$ is a sequence of Taylor-Maclaurin coefficients and $\mathfrak{m}_0$ and ϖ are constants and depend upon the bounded sequence $\{\kappa_n\}_{n\epsilon N_0}$. Corresponding to the function $\Theta\left(\{\kappa_n\}_{n\epsilon N_0}; z\right)$, Srivastava et al. (2012) defined extended Gamma function, extended Beta function and extended Gauss hypergeometric function respectively as

$$\Gamma_p^{\{\kappa_n\}_{n\epsilon N_0}}(z) = \int_0^{\infty} x^{z-1}\, \Theta\left(\{\kappa_n\}_{n\epsilon N_0}; -x - \frac{p}{x}\right) dx$$
$$(\Re(z) > 0; \Re(p) \geq 0)$$
$$B_p^{\left(\{\kappa_n\}_{n\epsilon N_0}\right)}(\alpha, \beta; p) = \int_0^1 x^{\alpha-1}(1 - x)^{\beta-1}\, \Theta\left(\{\kappa_n\}_{n\epsilon N_0}; -\frac{p}{x(1-x)}\right) dx.$$
$$(\min\{\Re(\alpha), \Re(\beta)\} \geq 0; \Re(p) \geq 0)$$
$$\mathfrak{J}_p^{\left(\{\kappa_n\}_{n\epsilon N_0}\right)}(a, b; c; z) = \sum_{k=0}^{\infty} (a)_k \frac{B_p^{\left(\{\kappa_n\}_{n\epsilon N_0}\right)}(b+k, c-b; p)}{B(b, c-b)} \frac{z^k}{k!}$$
$$(|z| < 1; \Re(c) > \Re(b) > 0; \Re(p) \geq 0) \quad \mathfrak{J}$$

It is assumed that all the integrals existed.

Corresponding to the extended Beta function $B_p^{\left(\{\kappa_n\}_{n\epsilon N_0}\right)}$, Parmar (2015) defined extension of Mittag-Leffler function

$$E_{\xi,\varsigma}^{\left(\{\kappa_n\}_{n\epsilon N_0}; \gamma\right)}(z; p) = \sum_{k=0}^{\infty} \frac{B_p^{\left(\{\kappa_n\}_{n\epsilon N_0}\right)}(\gamma+k, 1-\gamma; p)}{B(\gamma, 1-\gamma)} \frac{z^k}{\Gamma(\xi k + \varsigma)} \quad (1)$$

where

$$\begin{pmatrix} z, \varsigma, \gamma \epsilon C; \Re(\xi) > 0, \\ \Re(\varsigma) > 0, \Re(\gamma) > 1; p \geq 0 \end{pmatrix}$$

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https://doi.org/10.21833/ijaas.2018.10.005

It is noted that generalized and extended form of Mittag-Leffler function defined in literature by Desai et al. (2016), Kilbas et al. (2004), Mittag-Leffler (1903), Mittal et al. (2016), Nadir et al. (2014), Özarslan and Yilmaz (2014), and Rahman et al (2017a, 2017b) are special cases of the proposed function defined in (1). Some special cases of this function are descried as:

(i) When $\kappa_n = \frac{(\rho)_n}{(\sigma)_n}$ then the extended form of (1) takes the form

$$E_{\xi,\varsigma}^{(\rho,\sigma);\gamma}(z;p) = \sum_{k=0}^{\infty} \frac{B^{(\rho,\sigma)}(\gamma+k, 1-\gamma; p)}{B(\gamma, 1-\gamma)} \frac{z^k}{\Gamma(\xi k+\varsigma)}$$

under the condition

$$\begin{pmatrix} z,\varsigma,\gamma\epsilon C; \Re(\rho) > 0, \Re(\sigma) > 0 \\ \Re(\xi) > 0, \Re(\varsigma) > 0, \Re(\gamma) > 1; p \geq 0 \end{pmatrix}$$

(ii) If we select a bounded sequence $\kappa_n = 1$, then (1) reduces to the definition of Özarslan and Yilmaz (2014)

$$E_{\xi,\varsigma}^{\gamma}(z;p) = \sum_{k=0}^{\infty} \frac{\mathrm{B}(\gamma+k, 1-\gamma; p)}{B(\gamma, 1-\gamma)} \frac{z^k}{\Gamma(\xi k+\varsigma)}$$
$$\begin{pmatrix} z,\varsigma,\gamma\epsilon C; \Re(\xi) > 0, \\ \Re(\varsigma) > 0, \Re(\gamma) > 1; p \geq 0 \end{pmatrix}$$

(iii) Another special case of (1) is when $\kappa_n = 1$, and $p = 0$ then (1) reduces to Prabhakar's function (Prabhakar, 1971) of three parameters.

$$E_{\xi,\varsigma}^{\gamma}(z;p) = \sum_{k=0}^{\infty} \frac{(\gamma)_k z^k}{\Gamma(\xi k+\varsigma)k!}$$
$$(\xi,\varsigma,\gamma\epsilon C; \Re(\xi) > 0, \Re(\varsigma) > 0)$$

(iv) If we set $\alpha = \beta = 1$ then our expressions for $E_{\xi,\varsigma}^{(\{\kappa_n\}_{n\epsilon N_0};\gamma)}, E_{\xi,\varsigma}^{(\rho,\sigma);\gamma}$ and $E_{\xi,\varsigma}^{\gamma}$ reduces to the extended confluent hypergeometric functions

$$E_{1,1}^{(\{\kappa_n\}_{n\epsilon N_0};\gamma)}(z;p) = \phi_p^{(\{\kappa_n\}_{n\epsilon N_0})}(\gamma;1;z) \quad (2)$$
$$E_{1,1}^{((p,q);\gamma)}(z;p) = \phi_p^{(p,q)}(\gamma;1;z) \quad (3)$$
$$E_{1,1}^{\gamma}(z;p) = \phi_p(\gamma;1;z) \quad (4)$$

In order to establish our main results, we need definition of Fox-Wright function and the concept of Hadamard products.

Definition: As indicated by Pohlen (2009), Let $g(z) \coloneqq \sum_{k=0}^{\infty} x_k z^k$ and $h(z) \coloneqq \sum_{k=0}^{\infty} y_k z^k$ be two power series then the Hadamard product of power series is defined as

$$(g * h)(z) \coloneqq \sum_{k=0}^{\infty} x_k y_k z^k = (h \cdot g)(z) \quad (5)$$
$$(|z| < R)$$

where

$$R = \lim_{k\to\infty} \left|\frac{x_k y_k}{x_{k+1} y_{k+1}}\right|$$
$$= \left(\lim_{k\to\infty} \left|\frac{x_k}{x_{k+1}}\right|\right) \cdot \left(\lim_{k\to\infty} \left|\frac{y_k}{y_{k+1}}\right|\right) = R_g \cdot R_h$$

where R_g and R_h are the radii of convergence of two series $g(z)$ and $h(z)$ respectively. Therefore, in general, $R \geq R_g \cdot R_h$.

It is to be noted that if one of the power series is an analytical function, then the Hadamard product series is also an analytical function.

Definition: As considered by Samko et al. (1993), the generalized Wright's function is defined as follows

$${}_p\Psi_q \left[\begin{matrix}(A_1,\alpha_1),\dots,(A_p,\alpha_p); \\ (B_1,\beta_1),\dots,(B_q,\beta_q);\end{matrix} z\right]$$
$$= \sum_{k=0}^{\infty} \frac{\prod_{i=1}^{p} \Gamma((A_i + k\alpha_i)}{\prod_{i=1}^{q} \Gamma(B_i + k\beta_i)} \frac{z^k}{k!}$$

where the coefficients $\alpha_1, \dots, \alpha_p \epsilon \Re^+$ and $\beta_1, \dots, \beta_q \epsilon \Re^+$ with $1 + \sum_{i=1}^{q} \beta_i - \sum_{i=1}^{p} \alpha_i \geq 0$.

2. Caputo-type MSM fractional derivative formula for extended Mittag-leffler function

Here, in this section, our main object is to establish composition of new fractional derivative formulas so called Caputo-type Marichev-Saigo–Maeda fractional operator involving the extended Mittag-leffler function (1) which is defined by Parmar (2015). Some special cases of our main result are considered. We obtain our main goal by applying the Caputo-type MSM fractional derivative given in (8) and (9) to the proposed function (1). Thus, for this purpose, we need to recall the pair of fractional derivatives $D_{0+,\mathfrak{C}}^{\omega,\omega',\varepsilon,\varepsilon',\eta}$ and $D_{0-,\mathfrak{C}}^{\omega,\omega',\varepsilon,\varepsilon',\eta}$ which are defined in terms of the corresponding pairs of fractional integral operator $I_{0+}^{\omega,\omega',\varepsilon,\varepsilon',\eta}$ and $I_{0-}^{\omega,\omega',\varepsilon,\varepsilon',\eta}$ containing Appell function F_3 as a kernel.

Definition: As shown by Saigo and Maeda (1998), Let $\omega, \omega', \varepsilon, \varepsilon', \eta \epsilon C$ and the left-sided and right-sided Marichev-Saigo-Maeda fractional integral operators containing Appell function F_3 in their kernel are defined as

$$\left(I_{0+}^{\omega,\omega',\varepsilon,\varepsilon',\eta} f\right)(x) = \frac{x^{-\omega}}{\Gamma(\eta)} \int_0^{\infty} (x-t)^{\eta-1} t^{-\omega'} \cdot$$
$$F_3\left(\omega,\omega',\varepsilon,\varepsilon';\eta; 1-\frac{t}{x}, 1-\frac{x}{t}\right) f(t)dt \quad (6)$$

and

$$\left(I_{0-}^{\omega,\omega',\varepsilon,\varepsilon',\eta} f\right)(x) = \frac{x^{-\omega'}}{\Gamma(\eta)} \int_0^{\infty} (t-x)^{\eta-1} t^{-\omega} \cdot$$
$$F_3\left(\omega,\omega',\varepsilon,\varepsilon';\eta; 1-\frac{x}{t}, 1-\frac{t}{x}\right) f(t)dt \quad (7)$$

Let $\omega, \omega^{/}, \varepsilon, \varepsilon^{/}, \eta \epsilon C$ and the left-sided and right-sided Caputo-type MSM fractional differential operators containing Appell function in their kernel are defined as

$$\left(D_{0+,\mathfrak{E}}^{\omega,\omega^{/},\varepsilon,\varepsilon^{/},\eta} f\right)(x) = \left(I_{0+}^{-\omega^{/}-\omega,-\varepsilon^{/},-\varepsilon+[\Re(\eta)]+1,-\eta+[\Re(\eta)+1]} f^{([\Re(\eta)+1])}\right)(x) \qquad (8)$$

and

$$\left(D_{0-,\mathfrak{E}}^{\omega,\omega^{-},\varepsilon,\varepsilon^{-},\eta} f\right)(x)\ (-1)^{[\Re(\eta)]+1} \left(I_{0+}^{-\omega^{/}-\omega,-\varepsilon^{/},-\varepsilon+[\Re(\eta)]+1,-\eta+[\Re(\eta)+1]} f^{([\Re(\eta)+1])}\right)(x) \qquad (9)$$

Here, we discuss lemma, which is essential for the establishment of our main results. This lemma provides the image of power function $t^{\rho-1}$with Caputo-type Marichev-Saigo-Maeda fractional differentiation.

Lemma: Let $\omega, \omega^{/}, \varepsilon, \varepsilon^{/}, \eta, \rho \epsilon C$ and $m = [\Re(\eta)] + 1$

(a)Under condition

$$\Re(\rho) - m > max\begin{Bmatrix} 0, \Re(-\omega+\varepsilon) \\ \Re(-\omega-\omega^{/}-\varepsilon^{/}+\eta) \end{Bmatrix}$$

then image will be

$$\left(D_{0+,\mathfrak{E}}^{\omega,\omega^{/},\varepsilon,\varepsilon^{/},\eta} t^{\rho-1}\right)(x) = \frac{\Gamma(\rho)\Gamma(\rho-\varepsilon+\omega-m)}{\Gamma(-\varepsilon+\rho-m)\Gamma(\omega+\omega^{/}-\eta+\rho)} \cdot \frac{\Gamma(\omega+\omega^{/}+\varepsilon^{/}-\eta+\rho-m)}{\Gamma(\omega+\varepsilon^{/}-\eta+\rho-m)} x^{\omega+\omega^{/}-\eta+\rho-1}$$

(b) If condition

$$\Re(\rho) + m > max\begin{Bmatrix} \Re(-\varepsilon^{/}), \Re(\omega^{/}+\varepsilon-\eta), \\ \Re(\omega+\omega^{/}-\eta) + \Re(\eta) + 1 \end{Bmatrix}$$

is satisfied then image will be

$$\left(D_{0-,\mathfrak{E}}^{\omega,\omega^{/},\varepsilon,\varepsilon^{/},\eta} t^{\rho-1}\right)(x) = \frac{\Gamma(1+\varepsilon^{/}-\rho+m)\Gamma(1-\omega-\omega^{/}+\eta-\rho)}{\Gamma(1-\rho)\Gamma(1-\omega^{/}+\varepsilon^{/}-\rho+m)} \cdot \frac{\Gamma(1-\omega^{/}-\varepsilon+\eta-\rho+m)}{\Gamma(1-\omega-\omega^{/}-\varepsilon+\eta-\rho+m)} x^{\omega+\omega^{/}-\eta+\rho-1}$$

Proof: (a) Now using the definition of left-sided Caputo-type MSM fractional differentiation operator, we have

$$\left(D_{0+,\mathfrak{E}}^{\omega,\omega^{/},\varepsilon,\varepsilon^{/},\eta} t^{\rho-1}\right)(x) = \left(I_{0+}^{-\omega^{/},-\omega,-\varepsilon^{/}+m,-\varepsilon,-\eta+m} \frac{d^m}{dt^m} t^{\rho-1}\right)(x) = \frac{\Gamma(\rho)}{\Gamma(\rho-m)}\left(I_{0+}^{-\omega^{/},-\omega,-\varepsilon^{/}+m,-\varepsilon,-\eta+m} t^{\rho-m-1}\right)(x)$$

Thus using the results about MSM integration from Saigo and Maeda (1998) that is

$$\left(I_{0+}^{\omega,\omega^{/},\varepsilon,\varepsilon^{/},\eta} t^{\rho-1}\right)(x) = \frac{\Gamma(\rho)\Gamma(-\omega^{/}+\varepsilon^{/}+\rho)}{\Gamma(\varepsilon^{/}+\rho)\Gamma(-\omega-\omega^{/}+\eta+\rho)} \cdot \frac{\Gamma(-\omega-\omega^{/}+\varepsilon+\eta+\rho)}{\Gamma(-\omega^{/}-\varepsilon+\eta+\rho)} x^{-\omega-\omega^{/}+\eta+\rho-1}$$

then we reach the required result.

(b) Analogous to above, using right-sided Caputo-type MSM differential operator, we have

$$\left(D_{0-,\mathfrak{E}}^{\omega,\omega^{/},\varepsilon,\varepsilon^{/},\eta} t^{\rho-1}\right)(x) = (-1)^m\left(I_{0-}^{-\omega^{/},-\omega,-\epsilon^{/},-\varepsilon+m,-\eta+m} \frac{d^m}{dt^m} t^{\rho-1}\right)(x) = \frac{\Gamma(1-\rho+m)}{\Gamma(1-\rho)}\left(I_{0+}^{-\omega^{/},-\omega,-\epsilon^{/},-\varepsilon+m,-\eta+m} t^{\rho-m-1}\right)(x)$$

similarly, right-sided MSM integration result from Saigo and Maeda (1998) which is

$$\left(I_{0-}^{\omega,\omega^{/},\varepsilon,\varepsilon^{/},\eta} t^{\rho-1}\right)(x) = \frac{\Gamma(1-\rho-\varepsilon)\Gamma(1+\omega+\omega^{/}-\eta-\rho)}{\Gamma(1-\rho)\Gamma(1+\omega-\varepsilon-\rho)} \cdot \frac{\Gamma(1+\omega+\varepsilon^{/}-\eta-\rho)}{\Gamma(1+\omega+\omega^{/}+\varepsilon^{/}-\eta-\rho)} x^{-\omega-\omega^{/}+\eta+\rho-1}$$

we approach the required result.

In main theorem, part (a) deals with the left-hand sided Caputo-type MSM fractional derivative and part (b) deals with the right-hand sided Caputo – type MSM fractional derivative of the extended Mittag-Leffler function (1).

Theorem: (a) Let $\omega, \omega^{/}, \varepsilon, \varepsilon^{/}, \eta, \rho \epsilon C, m = [\Re(\eta)] + 1$ be such that

$$\Re(\rho) - m > max\begin{Bmatrix} 0, \Re(-\omega+\varepsilon) \\ \Re(-\omega-\omega^{/}-\varepsilon^{/}+\eta) \end{Bmatrix}$$

then

$$\left(D_{0+,\mathfrak{E}}^{\omega,\omega^{/},\varepsilon,\varepsilon^{/},\eta}\left(t^{\rho-1} E_{\xi,\varsigma}^{(\{\kappa_n\}_{n\epsilon N_0};\gamma)}(xt^{\sigma})\right)\right)(z) = z^{\omega+\omega^{/}-\eta+\rho-1} \cdot E_{\xi,\varsigma}^{(\{\kappa_n\}_{n\epsilon N_0};\gamma)}(xz^{\sigma}) *_4 \Psi_3\left[\begin{matrix}\Delta \\ \Delta^{/}\end{matrix}; xz^{\sigma}\right]$$

where

$$\Delta = \begin{Bmatrix} (\rho,\sigma), (\omega-\varepsilon+\rho-m,\sigma), \\ (\omega+\omega^{/}+\varepsilon^{/}-\eta+\rho-m,\sigma), (1,\sigma) \end{Bmatrix}$$

$$\Delta^{/} = \begin{Bmatrix} (-\varepsilon+\rho-m,\sigma), (\omega+\omega^{/}-\eta+\rho,\sigma), \\ (\omega+\varepsilon^{/}-\eta+\rho-m,\sigma) \end{Bmatrix}$$

b) Let $\omega, \omega^{/}, \varepsilon, \varepsilon^{/}, \eta, \rho \epsilon C, m = [\Re(\eta)] + 1$ be such that

$$\Re(\rho) + m > max\begin{Bmatrix} \Re(-\varepsilon^{/}), \Re(\omega^{/}+\varepsilon-\eta), \\ \Re(\omega+\omega^{/}-\eta) + m \end{Bmatrix}$$

then

$$\left(D_{0-,\mathfrak{E}}^{\omega,\omega',\varepsilon,\varepsilon',\eta}\left(t^{\rho-1}E_{\xi,\varsigma}^{(\{\kappa_n\}_{n\epsilon N_0};\gamma)}(xt^{-\sigma})\right)\right)(z)$$
$$= z^{\omega+\omega'-\eta+\rho-1}$$
$$\cdot E_{\xi,\varsigma}^{(\{\kappa_n\}_{n\epsilon N_0};\gamma)}(xz^{-\sigma}) *_4 \Psi_3\left[\begin{matrix}\Omega\\ \Omega'\end{matrix}; xz^{-\sigma}\right]$$

where

$$\Omega = \left\{\begin{matrix}(1+\varepsilon'-\rho+m,\sigma),(1-\omega-\omega'+\eta-\rho,\sigma)\\ (1-\omega'-\varepsilon+\eta-\rho+m,\sigma),(1,\sigma)\end{matrix}\right\}$$
$$\Omega' = \left\{\begin{matrix}(1-\rho,\sigma),(1-\omega'+\varepsilon'-\rho+m,\sigma),\\ (1-\omega-\omega'-\varepsilon+\eta-\rho+m,\sigma)\end{matrix}\right\}$$

Proof: Using the proposed function (1) and the definition of $\left(D_{0+,\mathfrak{E}}^{\omega,\omega',\varepsilon,\varepsilon',\eta}f\right)(x)$, and due to uniform convergence, order of summation and integration is changeable. Thus, we get

$$\left(D_{0+,\mathfrak{E}}^{\omega,\omega',\varepsilon,\varepsilon',\eta}\left(t^{\rho-1}E_{\xi,\varsigma}^{(\{\kappa_n\}_{n\epsilon N_0};\gamma)}(xt^{\sigma})\right)\right)(z)$$
$$=\sum_{k=0}^{\infty}\frac{B_p^{(\{\kappa_n\}_{n\epsilon N_0})}(\gamma+k,1-\gamma;p)}{B(\gamma,1-\gamma)}\frac{x^k}{\Gamma(\xi k+\varsigma)}$$
$$\cdot\left(D_{0+,\mathfrak{E}}^{\omega,\omega',\varepsilon,\varepsilon',\eta}t^{\rho+\sigma k-1}\right)(x)$$

thus, using the result of the lemma part (a) and replacing ρ by $\rho+\sigma k$, we have

$$\left(D_{0+,\mathfrak{E}}^{\omega,\omega',\varepsilon,\varepsilon',\eta}\left(t^{\rho-1}E_{\xi,\varsigma}^{(\{\kappa_n\}_{n\epsilon N_0};\gamma)}(xt^{\sigma})\right)\right)(z)$$
$$=\sum_{k=0}^{\infty}\frac{B_p^{(\{\kappa_n\}_{n\epsilon N_0})}(\gamma+k,1-\gamma;p)}{B(\gamma,1-\gamma)}\frac{x^k}{\Gamma(\xi k+\varsigma)}$$
$$\cdot\frac{\Gamma(\rho+\sigma k)\Gamma(\rho-\varepsilon+\omega+\sigma k-m)}{\Gamma(-\varepsilon+\rho+\sigma k-m)\Gamma(\omega+\omega'-\eta+\rho+\sigma k)}$$
$$\cdot\frac{\Gamma(\omega+\omega'+\varepsilon'-\eta+\rho+\sigma k-m)}{\Gamma(\omega+\varepsilon'-\eta+\rho+\sigma k-m)}\cdot z^{\omega+\omega'-\eta+\rho+\sigma k-1}$$

The last expression can easily be emerged by using Hadamard product rule given in Pohlen (2009) and we get the result.

(b) Analogously to the proof of above part, our demonstration of Caputo fractional derivative formula, depends upon the definition of the function (1) and the known result part (b) of the lemma, we have

$$\left(D_{0-,\mathfrak{E}}^{\omega,\omega',\varepsilon,\varepsilon',\eta}\left(t^{\rho-1}E_{\xi,\varsigma}^{(\{\kappa_n\}_{n\epsilon N_0};\gamma)}(xt^{-\sigma})\right)\right)(z)$$
$$=\sum_{k=0}^{\infty}\frac{B_p^{(\{\kappa_n\}_{n\epsilon N_0})}(\gamma+k,1-\gamma;p)}{B(\gamma,1-\gamma)}\frac{x^k}{\Gamma(\xi k+\varsigma)}$$

$$D_{0-,\mathfrak{E}}^{\omega,\omega',\varepsilon,\varepsilon',\eta}t^{\rho-\sigma k-1}$$

Now using the result of the lemma part (b) and changing ρ by $\rho-\sigma k$, we have

$$\left(D_{0-,\mathfrak{E}}^{\omega,\omega',\varepsilon,\varepsilon',\eta}\left(t^{\rho-1}E_{\xi,\varsigma}^{(\{\kappa_n\}_{n\epsilon N_0};\gamma)}(xt^{-\sigma})\right)\right)(z)$$
$$=\sum_{k=0}^{\infty}\frac{B_p^{(\{\kappa_n\}_{n\epsilon N_0})}(\gamma+k,1-\gamma;p)}{B(\gamma,1-\gamma)}\frac{x^k}{\Gamma(\xi k+\varsigma)}$$
$$=\frac{\Gamma(1+\varepsilon'-\rho+\sigma k+m)\Gamma(1-\omega-\omega'+\eta-\rho+\sigma k)}{\Gamma(1-\rho+\sigma k)\Gamma(1-\omega'+\varepsilon'-\rho+\sigma k+m)}$$
$$\cdot\frac{\Gamma(1-\omega'-\varepsilon+\eta-\rho+\sigma k+m)}{\Gamma(1-\omega-\omega'-\varepsilon+\eta-\rho+\sigma k+m)}$$
$$\cdot z^{\omega+\omega'-\eta+\rho-\sigma k-1}$$

Thus, with the help of a Hadmard product, last continuation expression converted into the required expression.

Number of formations of various types of Mittag-Leffler function depends upon parameters. Our main results can be deduced into numerous fractional calculus results defined by many authors. Setting $\kappa_n \to 1$the main Theorem yields the following result and the proposed function (1) reduced to the definition of Özarslan and Yilmaz (2014).

Corollary: Let the parameters $\omega,\omega',\varepsilon,\varepsilon',\eta,\rho\epsilon C$, $m=[\Re(\eta)]+1$ and under the stated conditions the left and the right-sided Caputo fractional differential operators of extended Mittag-Leffler function defined by Özarslan and Yilmaz (2014) is given below

$$\left(D_{0+,\mathfrak{E}}^{\omega,\omega',\varepsilon,\varepsilon',\eta}\left(t^{\rho-1}E_{\xi,\varsigma}^{\gamma}(xt^{\sigma})\right)\right)(z)$$
$$= z^{\omega+\omega'-\eta+\rho-1}\cdot E_{\xi,\varsigma}^{\gamma}(xz^{\sigma}) *_4 \Psi_3\left[\begin{matrix}\Delta_1\\ \Delta_1'\end{matrix}; xz^{\sigma}\right]$$

where

$$\Delta_1=\left\{\begin{matrix}(\rho,\sigma),(\omega-\varepsilon+\rho-m,\sigma),\\ (\omega+\omega'+\varepsilon'-\eta+\rho-m,\sigma),(1,\sigma)\end{matrix}\right\}$$
$$\Delta_1'=\left\{\begin{matrix}(-\varepsilon+\rho-m,\sigma),(\omega+\omega'-\eta+\rho,\sigma),\\ (\omega+\varepsilon'-\eta+\rho-m,\sigma)\end{matrix}\right\}$$

and

$$\left(D_{0-,\mathfrak{E}}^{\omega,\omega',\varepsilon,\varepsilon',\eta}\left(t^{\rho-1}E_{\xi,\varsigma}^{\gamma}(xt^{-\sigma})\right)\right)(z)$$
$$= z^{\omega+\omega'-\eta+\rho-1}\cdot E_{\xi,\varsigma}^{\gamma}(xz^{-\sigma}) *_4 \Psi_3\left[\begin{matrix}\Omega_1\\ \Omega_1'\end{matrix}; xz^{-\sigma}\right]$$

where

$$\Omega_1=\left\{\begin{matrix}(1+\varepsilon'-\rho+m,\sigma),(1-\omega-\omega'+\eta-\rho,\sigma),\\ (1-\omega'-\varepsilon+\eta-\rho+m,\sigma),(1,\sigma)\end{matrix}\right\}$$
$$\Omega_1'=\left\{\begin{matrix}(1-\rho,\sigma),(1-\omega'+\varepsilon'-\rho+m,\sigma),\\ (1-\omega-\omega'-\varepsilon+\eta-\rho+m,\sigma)\end{matrix}\right\}$$

It is to be noted that for suitable selection of $\kappa_n$numerous fractional calculus results can be deduced for number of types of Mittag-Leffler function defined in literature.

Further, if we replace $\xi=\varsigma=1$ then extensions of Mittag-Leffler function can be expressed in terms of the extended confluent hypergeometric functions (2), (3) and (4). Then we have the following relation:

Corollary:

$$\left(D_{0+,\mathfrak{E}}^{\omega,\omega',\varepsilon,\varepsilon',\eta}\left(t^{\rho-1}\phi_p^{(\{\kappa_n\}_{n\epsilon N_0})}(\gamma;1;xt^{\sigma})\right)\right)(z)$$
$$= z^{\omega+\omega'-\eta+\rho-1}\cdot\phi_p^{(\{\kappa_n\}_{n\epsilon N_0})}(\gamma;1;xz^{\sigma}) *_4 \Psi_3\left[\begin{matrix}\Delta_2\\ \Delta_2'\end{matrix};xz^{\sigma}\right]$$

where

$$\Delta_2=\left\{\begin{matrix}(\rho,\sigma),(\omega-\varepsilon+\rho-m,\sigma),\\(\omega+\omega'+\varepsilon'-\eta+\rho-m,\sigma),(1,\sigma)\end{matrix}\right\}$$
$$\Delta_2'=\left\{\begin{matrix}(-\varepsilon+\rho-m,\sigma),(\omega+\omega'-\eta+\rho,\sigma),\\(\omega+\varepsilon'-\eta+\rho-m,\sigma)\end{matrix}\right\}$$

and

$$\left(D_{0-,\mathfrak{E}}^{\omega,\omega',\varepsilon,\varepsilon',\eta}\left(t^{\rho-1}\phi_p^{(\{\kappa_n\}_{n\epsilon N_0})}(\gamma;1;xt^{-\sigma})\right)\right)(z)$$
$$= z^{\omega+\omega'-\eta+\rho-1}$$
$$\cdot\phi_p^{(\{\kappa_n\}_{n\epsilon N_0})}(\gamma;1;xz^{-\sigma}) *_4 \Psi_3\left[\begin{matrix}\Omega_2\\ \Omega_2'\end{matrix};xz^{-\sigma}\right]$$

where

$$\Omega_2=\left\{\begin{matrix}(1+\varepsilon'-\rho+m,\sigma),(1-\omega-\omega'+\eta-\rho,\sigma),\\(1-\omega'-\varepsilon+\eta-\rho+m,\sigma),(1,\sigma)\end{matrix}\right\}$$
$$\Omega_2'=\left\{\begin{matrix}(1-\rho,\sigma),(1-\omega'+\varepsilon'-\rho+m,\sigma),\\(1-\omega-\omega'-\varepsilon+\eta-\rho+m,\sigma)\end{matrix}\right\}$$

Remark: Several further consequences of our main result and above corollary (1) and (2) can easily be derived by selecting different values of the bounded sequence κ_n. If we set $\xi=\varsigma=1$ and $\kappa_n=0$ then the above results reduces for classical confluent hypergeometric functions. On replacing $\xi=\varsigma=1$ and $\kappa_n=1$, we get results for extended form of confluent hypergeometric function defined by Özarslan and Yilmaz (2014). Further, on setting $\xi=\varsigma=1$ and $\kappa_n=\frac{(\rho)_n}{(\sigma)_n}$ then the relation of the above corollary becomes for the confluent hypergeometric function defined by Chaudhry et al. (1997, 2004).

3. Further special cases

First time Rao et al. (2010) introduced and defined Caputo-type fractional derivative involving Gauss hypergeometric function in its kernel. Thus for $\omega,\varepsilon,\eta\epsilon C$ and $x\epsilon\mathfrak{R}^+$ with $\mathfrak{R}(\omega)>0$ Caputo fractional differentiation of Saigo's operators associated with Gauss hypergeometric function are defined as

$$\left(D_{0+,\mathfrak{E}}^{\omega,\varepsilon,\eta}f\right)(x)=$$
$$\left(I_{0+}^{-\omega+[\mathfrak{R}(\omega)]+1,-\varepsilon-[\mathfrak{R}(\omega)]-1,\omega+\eta-[\mathfrak{R}(\omega)]-1}f^{([\mathfrak{R}(\omega)+1])}\right)(x)$$

and

$$\left(D_{0-,\mathfrak{E}}^{\omega,,\varepsilon,\eta}f\right)(x)=(-1)^{[\mathfrak{R}(\omega)]+1}$$
$$\cdot\left(I_{0-}^{-\omega+[\mathfrak{R}(\omega)]+1,-\varepsilon-[\mathfrak{R}(\omega)]-1,\omega+\eta}f^{([\mathfrak{R}(\omega)+1])}\right)(x)$$

Remark: Caputo-type MSM fractional differential operators are associated with Appell function F_3 reduces to the Caputo fractional differential operators associated with hypergeometric function as follows having the following relationship:

$$\left(D_{0+,\mathfrak{E}}^{0,\omega',\varepsilon,\varepsilon',\eta}f\right)(z)=\left(D_{0+,\mathfrak{E}}^{\eta,\omega',-\eta,,\varepsilon'-\eta}f\right)(z)$$

and

$$\left(D_{0-,\mathfrak{E}}^{0,\omega',\varepsilon,\varepsilon',\eta}f\right)(z)=\left(D_{0-,\mathfrak{E}}^{\eta,\omega',-\eta,,\varepsilon'-\eta}f\right)(z)$$

Further on setting $\varepsilon\to 0$ in a Saigo Caputo differentiation operator reduces immediately to the Erdelyi-Kober fractional Caputo-type operator having the following relationship

$$D_{\eta,\omega}^{+,\mathfrak{E}}=D_{0+,\mathfrak{E}}^{\omega,0,\eta}\quad\text{and}\quad D_{\eta,\omega}^{-,\mathfrak{E}}=D_{0-,\mathfrak{E}}^{\omega,0,\eta}$$

Thus we get new fractional differential formulas of left- and right-sided Saigo and Erdelyi-Kober Caputo fractional differential operators stated in corollaries below.

Corollary: Let $\omega,\varepsilon,\eta,\rho\epsilon C, m=[\mathfrak{R}(\eta)]+1$ be such that $\mathfrak{R}(\rho)-m>max\{0,\mathfrak{R}(-\omega-\varepsilon-\eta)\}$. Then the left-hand sided generalized Caputo fractional differentiation $D_{0+,\mathfrak{E}}^{\omega,\varepsilon,\eta}$ of extended Mittag-leffler function is given by for $z>0$. Then we have the following relation:

$$\left(D_{0+,\mathfrak{E}}^{\omega,\varepsilon,\eta}\left(t^{\rho-1}E_{\xi,\varsigma}^{(\{\kappa_n\}_{n\epsilon N_0};\gamma)}(xt^{\sigma})\right)\right)(z)$$
$$= z^{\varepsilon+\rho-1}$$
$$\cdot E_{\xi,\varsigma}^{(\{\kappa_n\}_{n\epsilon N_0};\gamma)}(xz^{\sigma}) *_3 \Psi_2\left[\begin{matrix}\Delta_3\\ \Delta_3'\end{matrix};xz^{\sigma}\right]$$
$$\Delta_3=\{(\rho,\sigma),(\omega+\varepsilon+\eta+\rho-m,\sigma),(1,\sigma)\}$$
$$\Delta_3'=\{(\varepsilon+\rho,\sigma),(\eta+\rho-m,\sigma)\}$$

And under the condition $\omega,\varepsilon,\eta,\rho\epsilon C, m=[\mathfrak{R}(\eta)]+1$ Such that

$$\mathfrak{R}(\rho)+m>max\{\mathfrak{R}(\varepsilon)+m,\mathfrak{R}(-\omega-\eta)\}$$

the right-hand sided generalized Caputo fractional differentiation $D_{0-,\mathfrak{E}}^{\omega,,\varepsilon,\eta}$ of extended Mittag-leffler function is given by for $z>0$

$$\left(D_{0-,\mathfrak{E}}^{\omega,\varepsilon,\eta}\left(t^{\rho-1}E_{\xi,\varsigma}^{(\{\kappa_n\}_{n\epsilon N_0};\gamma)}(xt^{-\sigma})\right)\right)(z)$$
$$= z^{\varepsilon+\rho-1}$$
$$\cdot E_{\xi,\varsigma}^{(\{\kappa_n\}_{n\epsilon N_0};\gamma)}(xz^{-\sigma}) *_3 \Psi_2\left[\begin{matrix}\Omega_3\\ \Omega_3'\end{matrix};xz^{-\sigma}\right]$$

where

$$\Omega_3=\left\{\begin{matrix}(1-\varepsilon-\rho,\sigma),(1+\omega+\eta-\rho+m,\sigma),\\(1,\sigma)\end{matrix}\right\}$$
$$\Omega_3'=\{(1-\rho,\sigma),(1-\varepsilon+\eta-\rho+m,\sigma)\}$$

Corollary: Let $\omega,\varepsilon,\eta,\rho\epsilon C, m=[\mathfrak{R}(\eta)]+1$ be such that $\mathfrak{R}(\rho)-m>max\{0,\mathfrak{R}(-\omega-\eta)\}$ then the left-hand sided generalized Caputo-type Erdelyi-Kober fractional differentiation $D_{\eta,\omega}^{+,\mathfrak{E}}\left(=D_{0+,\mathfrak{E}}^{\omega,,0,\eta}\right)$ of extended Mittag-Leffler function is given by for $z>0$

$$D_{\eta,\omega}^{+,\mathfrak{E}}\left(t^{\rho-1}E_{\xi,\varsigma}^{(\{\kappa_n\}_{n\epsilon N_0};\gamma)}(xt^{\sigma})\right)(z)==z^{\rho-1}$$
$$\cdot E_{\xi,\varsigma}^{(\{\kappa_n\}_{n\epsilon N_0};\gamma)}(xz^{\sigma}) *_2\Psi_1\begin{bmatrix}\Delta_4\\ \Delta_4^{/};xz^{\sigma}\end{bmatrix}$$
$$\Delta_4=\{(\omega+\eta+\rho-m,\sigma),(1,\sigma)\}$$
$$\Delta_4^{/}=\{(\eta+\rho-m,\sigma)\}$$

and under the stated conditions $\omega,\varepsilon,\eta,\rho\epsilon C, m=[\Re(\eta)]+1$ such that $\Re(\rho)+m>max\{m,\Re(-\omega-\eta)\}$ then the right-hand sided generalized Caputo-type Erdelyi-Kober fractional differentiation $D_{\eta,\omega}^{-,\mathfrak{E}}\left(=D_{0-,\mathfrak{E}}^{\omega,,0,\eta}\right)$ of extended Mittag-leffler function is given by for $z>0$

$$D_{\eta,\omega}^{-,\mathfrak{E}}\left(t^{\rho-1}E_{\xi,\varsigma}^{(\{\kappa_n\}_{n\epsilon N_0};\gamma)}(xt^{-\sigma})\right)(z)==z^{\rho-1}$$
$$\cdot E_{\xi,\varsigma}^{(\{\kappa_n\}_{n\epsilon N_0};\gamma)}(xz^{-\sigma}) *_2\Psi_1\begin{bmatrix}\Omega_4\\ \Omega_4^{/};xz^{-\sigma}\end{bmatrix}$$

where

$$\Omega_4=\{(1+\omega+\eta-\rho+m,\sigma),(1,\sigma)\}$$
$$\Omega_4^{/}=\{(1+\eta-\rho+m,\sigma)\}$$

It is to be noted that several further consequences of the main Theorem and Corollaries 3-4 can easily be converted to many other known result by suitable substitutions of the parameters.

4. Conclusion

In this paper, we obtain the Caputo type MSM fractional derivative of family of Mittag-Leffler function. It is to be noted that said operator transform the required function into a function of higher order. Further, well known operators like Erdelyi-Kober and Saigo's operators are the special case of Marichev-Saigo-Maeda fractional operator.

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